MAT 3103: Statistics and Probability **Semester: Summer 2018**

Chapter 2: Probability and Events

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2.1 Sample Space, Event Space and Probability

Sample space is the set of all possible outcomes in an experiment or random trial. For example, in a cricket match a coin is tossed to decide for game plan (batting or fielding first). There are only two results, either head (H) or tail (T) can be observed. Thus there are only $n = 2$ possible outcomes and the sample space is $S = \{H, T\}$.

Event space is a subset of sample space. For example, coin tossing for a cricket match has two possible outcomes, H and T. Let us assume that a coin is tossed and head is observed. Then the event space is $E = \{H\}$. Evidently, $E \subset S$.

Probability of an event is the probability associated to the event space and is computed as

$$P(E) = \frac{\text{Cardinality of event space } E}{\text{Cardinality of sample space } S} = \frac{m}{n}$$

For coin tossing in a cricket match, $S = \{H, T\}$ and the cardinality (number of elements of a set) of S is $n = 2$. Similarly, the cardinality of event space $E = \{H\}$ is $m = 1$. Thus the probability that head (H) will be observed in a coin tossing is

$$P(E) = \frac{m}{n} = \frac{1}{2}$$

2.1.1 Construction of sample space

Sample space for tossing a coin is $S = \{H, T\}$ and its construction has been explained in the above section.

Sample space for two coins is constructed by writing sample space for one coin in the row and that of the other coin in the column. Thus sample space is a set of 4 elements shown below.

	H	T
H	HH	HT
T	TH	TT

Number of elements in the sample space (all possible outcome) is $n = 4$.

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Sample space for 3 coins: place sample space for two coins in the row and the other coin in the column. Number of elements in the sample space (all possible outcome) is $n = 8$.

	HH	HT
	TH	TT
H	HHH	HHT
	HTH	HTT
T	THH	THT
	TTH	TTT

2.1.2 Probability of an event

We construct an event set by using favorable outcomes. Then the probability of occurring or happening an event A is

$$P(A) = \frac{\text{Number of favorable outcomes}}{\text{Number of all possible outcomes}} = \frac{\text{Cardinality of set A}}{\text{Cardinality of set S}} = \frac{m}{n}$$

Example 1: A fair coin is tossed once. What is the probability that a head will be shown?

A fair coin means, the probability of observing head (H) equal to the probability of observing tail (T). A coin is tossed once, so the sample space is $S = \{H, T\}$. Let us now define the event that head will be shown. Thus the event set $A = \{H\}$ and

$$P(A) = \frac{m}{n} = \frac{1}{2}$$

Example 2: Two fair coins are tossed once. What is the probability that (i) both coins will show head (ii) at least one coin will show tail (iii) at most (or at best) one coin will show head, and (iv) none of the coins will show head?

A fair coin means, the probability of observing head (H) equal to the probability of observing tail (T). Sample space for two coins is $S = \{HH, HT, TH, TT\}$ and the cardinality is $m = 4$.

(i) Let us define the event as observing head (H) for both coins , that is, $A = \{HH\}$. Thus $m = 1$ and $P(A) = \frac{m}{n} = \frac{1}{4}$.

(ii) At least one coin will show tail (T), that is, the event set $B = \{HT, TH, TT\}$ and $m = 3$. Thus $P(B) = \frac{m}{n} = \frac{3}{4}$

(iii) At best one coin will show head, that is, $C = \{HT, TH, TT\}$ and $P(C) = \frac{3}{4}$

(iv) None of the coins will show head, that is, the event set $D = \{TT\}$ and $P(D) = \frac{1}{4}$

Example 3: Three fair coins are tossed once. What is the probability that (i) at least two coins will show head (ii) at most (or at best) two coins will show head, and (iii) first coin will show head (iv) first and third coin will show head (v) first or third coin will show head (vi) third coin shows head given the first coin shows head?

A fair coin means, the probability of observing head (H) equal to the probability of observing tail (T). Sample space for three coins is

$$S = \left\{ \begin{array}{ll} HHH & HHT \\ HTH & HTT \\ THH & THT \\ TTH & TTT \end{array} \right\}$$

and the cardinality is $n = 8$.

(i) Let A be an event that at least two coins will show head. So, $A = \{HHH, HHT, HTH, THH\}$. Thus $m = 4$ and $P(A) = \frac{m}{n} = \frac{4}{8} = \frac{1}{2}$.

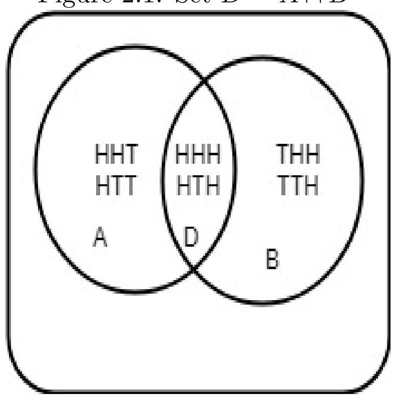
(ii) At most two coins will show head (H), that is, the event set $B = \{HHT, HTH, HTT, THH, THT, TTH, TTT\}$ and $m = 7$. Thus $P(B) = \frac{m}{n} = \frac{7}{8}$

(iii) The first coin will show head, that is, $C = \{HHH, HHT, HTH, HTT\}$ and $P(C) = \frac{4}{8}$

(iv) The first and the third coins will show head, that is, the event set $D = \{HHH, HTH\}$ and $P(D) = \frac{2}{8}$

(v) We can also show that the set $A = \{HHH, HHTHTH, HTT\}$ is the set of outcomes with first coin showing head (H) and the set $B = \{HHH, HTH, THH, TTH\}$ is the set of outcomes with the third coin showing head. Thus outcomes common to both sets are $A \cap B = \{HHH, HTH\}$, which an event set with the first and the third coins showing head. Thus the probability that both the first and the third coins show head is essentially $P(A \cap B)$. This can be clearer from the Venn diagram shown below.

Figure 2.1: Set $D = A \cap B$



We want to find the probability that the first or the third coin shows head, that is, we want to find $P(A \cup B)$.

$$
\begin{aligned}
P(A) &= \frac{4}{8} \\
P(B) &= \frac{4}{8} \\
P(A \cap B) &= \frac{2}{8} \\
P(A \cup B) &= P(A) + B - P(A \cap B) \\
&= \frac{4}{8} + \frac{4}{8} - \frac{2}{8}
\end{aligned}
$$

Note: For "and", please check $P(A \cap B)$, for "or" use $P(AUB)$

(vi) Third coin shows head given the first coin shows head. Thus we would like to find

$$P(\text{B given A}) = P(\text{B conditional on A}) = P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{2}{8}}{\frac{4}{8}} = \frac{2}{4} = \frac{1}{2}$$

2.2 Independent and Mutually Independent Events

Two events A and B are independent if

$$P(A \cap B) = P(A)P(B)$$

Let us recall the coin tossing problem in Example 3(v), where we have shown that $P(A) = \frac{4}{8}$, $P(B) = \frac{4}{8}$, and $P(A \cap B) = \frac{2}{8} = \frac{1}{4} = P(A)P(B)$. Thus the events A and B are independent.

When two events are independent, then one does not depend on the other. Thus, the conditional probability $P(B|A) = P(B)$, because B does not depend on A.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$$

In Example 3(v), we have found $P(B|A) = \frac{1}{2}$ and $P(B) = \frac{4}{8} = \frac{1}{2}$. Thus the event B does not depend on the event A.

Mutually independent event: If three events A, B and C are pairwise independent $P(AB) = P(A)P(B)$, $P(BC) = P(B)P(C)$ and $P(CA) = P(C)P(A)$) and $P(ABC) = P(A)P(B)P(C)$, then the events are mutually independent. Let us assume that $A = \{HHH, HHT, HTH, HTT\}$, $B = \{HHH, HHT, THH, THT\}$, and $C = \{HHH, HTH, THH, TTH\}$ are events denoting the first, second and the third coin shows head, respectively. Then $A \cap B = \{HHH, HHT\}$, $B \cap C = \{HHH, THH\}$, $C \cap A = \{HHH, HTH\}$, and $A \cap B \cap C = \{HHH\}$. Thus

$$\begin{aligned}
P(A) &= \frac{4}{8} \\
P(B) &= \frac{4}{8} \\
P(C) &= \frac{4}{8} \\
P(AB) = P(A \cap B) &= \frac{2}{8} = \frac{1}{4} = P(A)P(B) \\
P(BC) = P(B \cap C) &= \frac{2}{8} = \frac{1}{4} = P(B)P(C) \\
P(CA) = P(C \cap A) &= \frac{2}{8} = \frac{1}{4} = P(C)P(A) \\
P(ABC) = P(A \cap B \cap C) &= \frac{1}{8} = P(A)P(B)P(C)
\end{aligned}$$

Thus these three events A, B and C are mutually independent.

2.3 Probability When Sampling With/Without Replacement

If a random experiment shows n exhaustive, mutually exclusive and equally likely outcomes and if there are $m(\leq n)$ outcomes in favour of an event A, then the probability of an event A is

$$P(A) = \frac{\text{No of outcomes favourbale to event A}}{\text{Number of all possible outcomes in the experiment}} = \frac{m}{n}$$

where, $0 \leq P(A) \leq 1$.

Example 1: In an experiment, 12 patients are tested for antibiotic resistance. Among them 7 are diagnosed resistant to the drug X and 5 are resistant to drug Y. Two patients are re-examined one after another under three selection procedures (1) at random (2) one by one with replacement, (3) one by one without replacement. For each selection procedure, find the probability that (a) both are resistant to drug X (b) one is resistant to X and the other is resistant to Y.

Total number of patients are $7 + 5 = 12$. Total number of ways 2 patients can be selected is $\binom{12}{2} = 66$ ways.

(1) when at random then

(a) an event A representing both patients resistant to drug X, will have $P(A) = \frac{m}{n} = \frac{\binom{7}{2}}{\binom{12}{2}}$

(b) an event B representing one resistant to X and the other to Y, will have $P(B) = \frac{\binom{7}{1}\times\binom{5}{1}}{\binom{12}{2}}$

(2) one by one selection will yield a sample space for two patients as

	X	Y
X	XX	XY
Y	YX	YY

From the sample space $S = \{XX, XY, YX, YY\}$ we may note that the event $A = \{XX\}$ for both patients diagnosed with resistance to X and the event $B = \{XY, YX\}$ for one to be diagnosed with X and the other with Y resistance.

(a) $P(A) = P(XX) = \frac{\binom{7}{1}}{\binom{12}{1}} \times \frac{\binom{7}{1}}{\binom{12}{1}}$

(b) $P(B) = P(XY) + P(YX) = \frac{\binom{7}{1}}{\binom{12}{1}} \times \frac{\binom{5}{1}}{\binom{12}{1}} + \frac{\binom{5}{1}}{\binom{12}{1}} \times \frac{\binom{7}{1}}{\binom{12}{1}}$

(3) one by one without replacement

(a) $P(A) = P(XX) = \frac{\binom{7}{1}}{\binom{12}{1}} \times \frac{\binom{6}{1}}{\binom{11}{1}}$

(b) $P(B) = P(XY) + P(YX) = \frac{\binom{7}{1}}{\binom{12}{1}} \times \frac{\binom{5}{1}}{\binom{11}{1}} + \frac{\binom{5}{1}}{\binom{12}{1}} \times \frac{\binom{7}{1}}{\binom{11}{1}}$

Example 2: About 75% signals transmitted from a sensor are recorded properly by a receptor. Three signals are selected randomly to follow up whether these are recorded by the receptor. Find the probability that (i) all 3 are recorded, (ii) two of them are recorded, (iii) at least one of the signals is recorded, (iv) at best two are recorded by the receptor.

Let us denote the result recorded (R) and not recorded (N) by R and N respectively. There are three signals with two possible results for each of them. Thus the number of possible outcomes would be $2^3 = 8$ and the sample space would be $S = \{RRR, RRN, RNR, NRN, NRR, RNN, NNR, NNN\}$. It is also known that $P(R) = 0.75$ and $P(N) = 0.25$.

(i) Let A be the set of outcomes that all three are recorded, that is, $A = \{RRR\}$ and so $P(A) = P(RRR) = 0.75 \times 0.75 \times 0.75$

(ii) Let B be the set of outcomes that two of them are recorded, that is, $B = \{RRN, RNR, NRR\}$. Thus $P(B) = P(RRN) + P(RNR) + P(NRR) = 0.75 \times 0.75 \times 0.25 + 0.75 \times 0.25 \times 0.75 + 0.25 \times 0.75 \times 0.75$

(iii) Let C be the set of outcomes that at least one of them are recorded, that is,

$$C = \{RRR, RRN, RNR, NRN, NRR, RNN, NNR\}$$

. Thus

$$P(C) = P(RRR) + P(RRN) + P(RNR) + P(NRN) + P(NRR) + P(RNN) + P(NNR) = ?$$

(iv) Let D be the set of outcomes that at best two of them are recorded, that is,

$$D = \{RRN, RNR, NRN, NRR, RNN, NNR, NNN\}$$

Thus

$$P(D) = P(RRN) + P(RNR) + P(NRN) + P(NRR) + P(RNN) + P(NNR) + P(NNN) = ?$$

2.4 Probability from Outcome Matrix

In an experiment, deep sleep and mediation state of 20 patients are examined by analyzing delta and theta bands in EEG signals. Outcome of Delta (D) and Theta (T) bands are classified into Normal (N) and Abnormal (A). Experimental results are as in the following table.

	Bands		
Results	D	T	Total
N	6	6	12
A	5	3	8
Total	11	9	20

Calculate (i) the probability that a patient will be diagnosed normal (N) with delta band (D) and (ii) a patient will be diagnosed abnormal (A).

(i) Probability that a patient will be diagnosed normal delta band is $P(N \cap D) = \frac{m}{n} = \frac{6}{20}$

(ii) Probability that a patient will be diagnosed abnormal is $P(A) = \frac{8}{20}$

2.5 Baye's Theorem

Probability of event A when the event B has subsequently occurred is

$$P(A|B) = \frac{P(A)P(B|A)}{[P(A)P(B|A)] + [P(A^c)P(B|A^c)]}$$

where A^c is the complementary set of A.

Example 1: Let us assume that 49% of adults in a Dhaka city area are male. One survey revealed that 75% female and 60% males are active Facebook users. From a Facbook users database, one person is selected at random. What is the probability that the selected Facebook user is a male?

We may note that $P(M) = P(Male) = 0.49$, and $P(F) = P(Female) = P(M^c) = 1 - 0.49 = 0.51$. It is also found that 60% males are active facebook users, that is, $P(U|M) = 0.6$. On the other hand, 75% females are active facebook users, that is, $P(U|F) = P(U|M^c) = 0.75$. Then

$$\begin{aligned} P(M|U) &= \frac{P(M)P(U|M)}{[P(M)P(U|M)] + [P(M^c)P(U|M^c)]} \\ &= \frac{0.49 \times 0.6}{[0.49 \times 0.6] + [0.51 \times 0.75]} \end{aligned}$$

The above mentioned version is suitable only when there are two events. But for more than two events we must make sure that

(i) Events must be disjoint

(ii) Events are exhaustive (all events will form a set with all possible outcomes)

Let us assume that H_i for $i = 1, ..., k$ are k disjoint events and the event E has an intersection with event $E \cap H_i$ with event H_i. Thus we may write

$$E = (E \cap H_1) \cup \ldots \cup (E \cap H_k)$$

Then the Baye's theorem states that

$$P(H_i|E) = \frac{P(H_i)P(E|H_i)}{\sum_{i=1}^{k} P(H_i)P(E|H_i)}$$

2.5.1 Exercises

1. Two robots are bought to work together in a restaurant to perform three tasks: Welcoming (W), Order placing (O) and Kebab making (K).
   (i) Construct the sample space for assigning duties to two robots
   (ii) Compute the probability that at least one robot will be making kebab
   (iii) Compute the probability that the second robot makes kebab given the first robot welcomes the customer.

2. Two robots can perform three tasks welcoming customers (W), singing (S) and dancing (D). The robotic restaurant owner assigns different tasks to different robots, as a result both robot can not perform the same duty.
   (i) Construct the sample space for assigning duties to these robots
   (ii) Compute the probability of second robot dances given the first robot sings.

3. Signals are sent from Station-1 and Station-2. Fifty signals from Station-1 and 30 signals from Station-2 are sent daily. Among these signals 20% sent from Station-1 and 30% sent from Station-2 do not reach properly. In a random investigation it is found that one signal is not reached properly. Find the probability that the signal is sent from Station-2.

4. Two robots A and B receive telephone calls, communicate with customers and attract orders for a product. Robot A attracts orders successfully in 70% cases and robot B can do it successfully in 60% cases. If robots fail to attract a customer for an order, the call is transferred to the officer-in-charge. Find the probability that
   (i) a potential buyers order will be placed successfully
   (ii) robot A becomes successful under the condition that B fails
   (iii) both robots will fail
   (iv) call will be transferred to the officer-in-charge.

5. In a box there are 60% mechanical engineering books and 40% computer science books. Among computer science books 50% are foreign books and among mechanical engineering books 40% are foreign books. A foreign book is selected. What is the probability that the selected one is computer science book?