Course: Time Series Analysis and Forecasting **Semester(s): 2008 (updated 2013)**

Chapter 1: Autocovariance Function

Atikur R. Khan, University of Rajshahi

1.1 Autocovariance and autocorrelation functions

- Autocoriance function (ACF): $\gamma(k) = E[(x_t - \mu)(x_{t+k} - \mu)]$
- Autocorrelation function: $\rho(k) = \frac{\gamma(k)}{\gamma(0)}$
- If $\mu = E[x_t] = 0$ then $\gamma(k) = E[x_t x_{t+k}]$

In practice $\gamma(k)$ is not known and is estimated from the data.

- $\{x_1, x_2, \ldots, x_N\}$ be a realization of the process x_t of length N
- $\bar{x} = N^{-1} \sum_{t=1}^{N} x_t$
- $\hat{\gamma}(k) = \frac{1}{N} \sum_{t=k+1}^{N} (x_t - \bar{x})(x_{t-k} - \bar{x})$ for $k = 0, 1, \ldots, N-1$.
- $\hat{\rho}(k) = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)}$, $k = 0, 1, \ldots, N-1$.

1.1.1 Example 1: US Accidental death data

Figure 1.1 shows monthly US accidental death numbers. This data set is readily available in `R` and has been used in numerous books and articles. We would like to examine autocovariance property of this time series.

Figure 1.4 shows autocorrelation function ($\hat{\rho}_k, k = 0, 1, \ldots, 24$) of US accidental death data. A 95% confidence interval (CI) is also added to the plot. Not all autocorrelation are lying within the CI and are not close to zero. This indicates the data is not random, rather depicts a high degree of autocorrelation between adjacent and near adjacent observations. Also, spikes of autocorrelation functions appear to be in cyclical fashion with an indication of existence of strong seasonal pattern in the data.

[1]Prepared by Atikur R. Khan (atikrkhan@gmail.com).

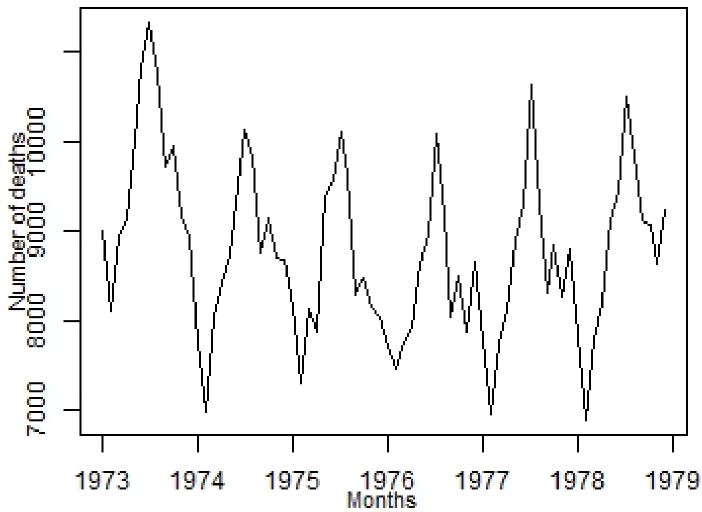

Figure 1.1: Monthly USA accidental death data

```
R code to produce Figure 1.1:

         plot.ts(USAccDeaths,xlab="",ylab="",main="")
         mtext("Number of deaths",side=2, cex=0.9, line=+1.75)
         mtext("Months", side=1, cex=0.9, line=+1.6)

R code to produce Figure 1.2:
        acf(USAccDeaths, lag.max=24, type="correlation", ci=0.95, xlab="k",
                  ylab="", main="", axes=FALSE)
        axis(1, at=c(0.0, 0.5, 1.0, 1.5, 2.0), label=c(0,6,12,18,24)  )
        axis(2)
        box()
        mtext(bquote( hat(rho[k]) ),side=2,cex=1.1,line=+1.75)
        mtext("k", side=1, cex=1, line=+1.6)
```

1.1.2 Example 2: Nottingham castle temperature data

This data set comprises monthly temperature data (in degrees Fahrenheit) of 20 years from January 1920 to December 1939. There is a clear seasonal pattern in the data (Figure 1.3) and we would expect autocorrelation function to show some spikes appear to be in cyclical patterns. Figure 1.4 shows that many autocorrelation functions are outside the 95% CI and there is a very strong autocorrelation in the data.

1.2 Covariance stationary process

A time series $\{x_t\}$ is covariance stationary if the mean and covariance function does not depend on time t so that

$$\begin{aligned} E(x_t) =& \mu \quad \text{for all } t \\ Cov(x_t, x_{t-k}) =& E\left[(x_t - \mu)(x_{t-k} - \mu)\right] = \gamma(k) \text{ for all } t \text{ and any } k \end{aligned}$$

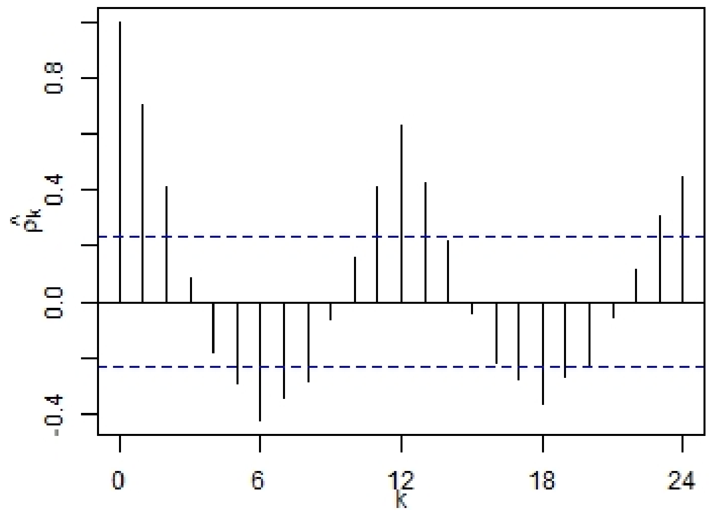

Figure 1.2: Sample autocorrelation function of monthly USA accidental death data

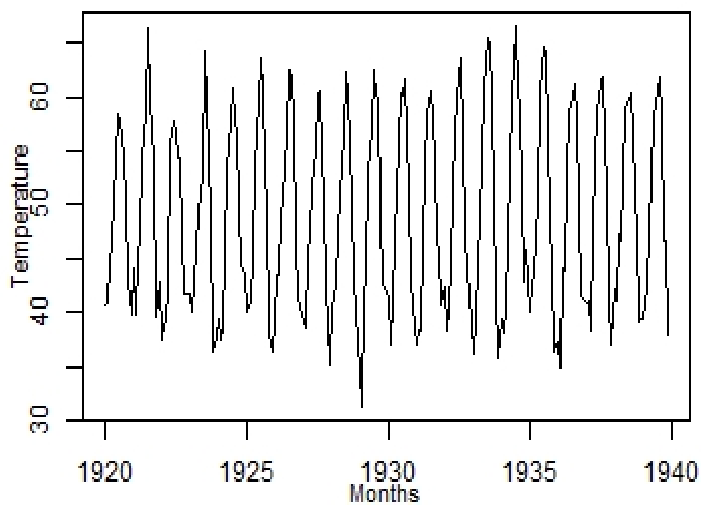

Figure 1.3: Monthly Nottingham castle temperature

1.2.1 Wold's representation

Wold's representation theorem, namely, that if x_t is a linearly regular, covariance-stationary process then x_t can be expressed as

$$x_t = \mu + \sum_{j=0}^{\infty} \kappa(j)\varepsilon_{t-j}\,, \tag{1.1}$$

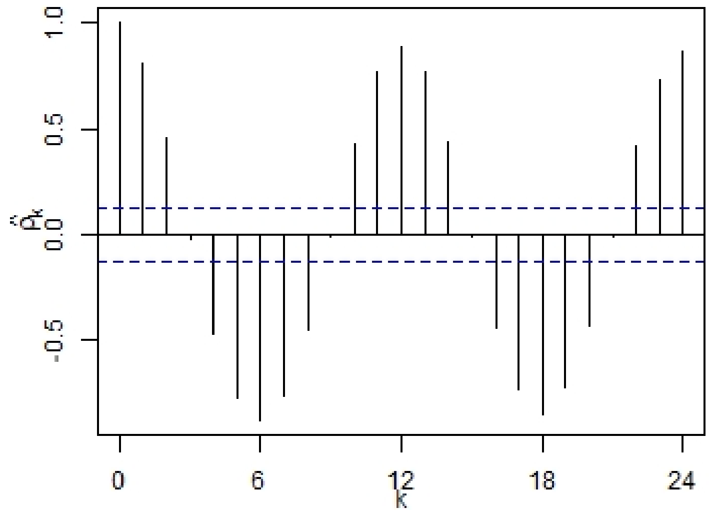

Figure 1.4: Sample autocorrelation function of monthly Nottingham castle temperature data

where ε_t is a zero mean white noise (innovation) process with variance σ^2. The coefficients of the transfer function $k(z) = \sum_{j\geq 0} \kappa(j)z^j$ satisfy the conditions $\kappa(0) = 1$ and $\sum_{j\geq 0} \kappa(j)^2 < \infty$. If $0 < \sum_{j\geq 0} |\kappa(j)| < \infty$ then x_t is said to be a short memory processes, whereas if $\sum_{j\geq 0} |\kappa(j)| = \infty$ then x_t is said to exhibit long memory, see Beran (1994) or Palma (2007). This division of linearly regular processes into short and long memory series according to the speed of decay of their impulse response coefficients proves to be of crucial importance in our subsequent analysis.

Definition 1.1. *A stationary process x_t is linearly regular (nonsingular) if and only if $\mathcal{X}_{-\infty} = \bigcap_{s=0}^{\infty} \mathcal{X}_{-\infty}^{t-s} = 0$ and linearly singular (deterministic) if and only if $\mathcal{X}_{-\infty} = \mathcal{X} = \bigcup_{s=-\infty}^{\infty} \mathcal{X}_{-\infty}^{t-s}$.*

From the argument presented in Ibragimov & Linnik (1971, Section 17.1) we can deduce the following result.

Theorem 1.1. *A necessary and sufficient condition for x_t to be a linearly regular, stationary process is that, for all $\zeta \in \mathcal{X}$ with $E[\zeta^2] < \infty$,*

$$\lim_{t\to-\infty} sup_\xi |E[\xi\zeta] - E[\xi]E[\zeta]| = 0\,,$$

where the supremum is taken over all $\xi \in \mathcal{X}_{-\infty}^t$ with $E[\xi^2] < \infty$.

Theorem 1.1 conveys the idea that for a linearly regular process events become uncorrelated (orthogonal) with increasing separation, and if we suppose that x_t for $t \in$ Z is a linearly regular, covariance-stationary process, with mean $E[x_t] = \mu$ and autocovariance function $E[(x_t - \mu)(x_{t+h} - \mu)] = \gamma(h)$, an obvious implication of the theorem is that $|\gamma(k)| \to 0$ as $k \to \infty$.

1.2.2 Example 1: White noise process

If $x_t = \varepsilon_t$ where $\varepsilon_t \sim WN(0, \sigma^2)$, then we get

$$\begin{aligned} E(x_t) =& E(\varepsilon_t) = 0 \\ Var(x_t) =& Var(\varepsilon_t) = \sigma^2 \\ Cov(x_t, x_{t-k}) =& Var(\varepsilon_t, \varepsilon_{t-k}) = 0 \end{aligned}$$

1.2.3 Example 2: Sinusoidal signal

Dealing with sinusoidal signal is very common in time series analysis especially in engineering and physics. Here we consider an example from Khan and Poskitt (2010) where cosine series is generated by using the formula

$$
\begin{aligned}
x_t =& \mu + \sum_{r=1}^{p} A_r cos(\lambda_r t + \theta_r) \\
y_t =& x_t + \varepsilon_t
\end{aligned}
$$

where μ is the mean signal, A_r is the amplitude, λ_r is the frequency (number of cycles per unit measured in radians), θ_r is the phase uniformly distributed over the range $(-\pi, \pi)$ and ε_t is the i.i.d. Gaussian noise process with noise variance σ^2. The wavelength of the process is $2\pi/\lambda_r$.

Since amplitude A_r and the frequency λ_r are constant at time t, we may write

$$E(x_t) = \mu + \sum_{r=1}^{p} A_r E\left[cos(\lambda_r t + \theta_r)\right] = \mu + \sum_{r=1}^{p} A_r \int_{-\pi}^{\pi} \frac{cos(\lambda_r t + \theta_r)}{2\pi} d\theta_r = \mu$$

Similarly, we can show that

$$E(x_t x_{t+h}) = \begin{cases} \mu^2 + \frac{1}{2}\sum_{r=1}^{p} A_r^2 cos(\lambda_r h) & \text{if } h \neq 0; \\ \mu^2 + \frac{1}{2}\sum_{r=1}^{p} A_r^2 & \text{if } h = 0. \end{cases}$$

and

$$E(y_t y_{t+h}) = \begin{cases} \mu^2 + \frac{1}{2}\sum_{r=1}^{p} A_r^2 cos(\lambda_r h) & \text{if } h \neq 0; \\ \mu^2 + \frac{1}{2}\sum_{r=1}^{p} A_r^2 + \sigma^2 & \text{if } h = 0. \end{cases}$$

Thus the process is covariance stationary and the signal-to-noise ratio of this process can be written as

$$SNR = 10\log_{10}\left(\frac{\mu^2 + \frac{1}{2}\sum_{r=1}^{p} A_r^2}{\sigma^2}\right) dB.$$

1.3 Properties of sample autocorrelation function

Let $x_1, x_2, \ldots, x_N$ be a sample from the time series $\{x_t : t \in T\}$ with finite mean μ and variance σ_x^2 then the hth sample autocorrelation function is

$$\hat{\gamma}(h) = \begin{cases} N^{-1}\sum_{t=1}^{N-|h|}(x_t - \bar{x})(x_{t+|h|} - \bar{x}) & \text{if } \mu \neq 0; \\ N^{-1}\sum_{t=1}^{N-|h|} x_t x_{t+|h|} & \text{if } \mu = 0. \end{cases}$$

1.3.1 Asymptotic mean and variance

Lemma 1.1. *If $\{x_t\}$ be a time series with mean $\mu = Ex_t < \infty$ and variance $\sigma_x^2 = var(x_t) < \infty$ then*

$$E\hat{\gamma}(h) = \begin{cases} \left(1 - \frac{|h|}{N}\right)\gamma(h) & \textit{if } \mu = 0; \\ \left(1 - \frac{|h|}{N}\right)\left[\gamma(h) - N^{-1}\gamma(0)\sum_{r=-(N-1)}^{N-1}(1 - |r|/N)\rho(r)\right] & \textit{if } \mu \neq 0. \end{cases}$$

where $\hat{\mu} = N^{-1}\sum_{t=1}^{N} x_t$ and $f(0)$ is the spectral density at frequency zero.

Proof of Lemma 1.1: When $\mu = 0$:

$$\begin{aligned} E\hat{\gamma}(h) &= N^{-1} \sum_{t=1}^{N-|h|} E x_t x_{t+|h|} \\ &= N^{-1}(N-|h|)\gamma(h) \\ &= \left(1 - \frac{|h|}{N}\right)\gamma(h) \end{aligned}$$

.

When $\mu \neq 0$:

$$\begin{aligned} E\hat{\gamma}(h) &= N^{-1} \sum_{t=1}^{N-|h|} E(x_t - \bar{x})(x_{t+|h|} - \bar{x}) \\ &= N^{-1} \sum_{t=1}^{N-|h|} E\left[(x_t - \mu)(x_{t+|h|} - \mu) - (\bar{x} - \mu)^2\right] \\ &= N^{-1} \sum_{t=1}^{N-|h|} \left[\gamma(h) - var(\bar{x})\right] \end{aligned}$$

.

Now $var(\bar{x}) \sim \frac{2\pi\gamma(0)}{N} f(0)$ yields $E\hat{\gamma}(h) \sim \left(1 - \frac{|h|}{N}\right)\left[\gamma(h) - \frac{2\pi\gamma(0)}{N} f(0)\right]$, which completes the proof.

Lemma 1.2. *If $\{x_t\}$ is a zero mean stationary process and is stationary up to order four then*

(i) $E\hat{\gamma}(h) = \left(1 - \frac{h}{N}\right)\gamma(h)$ *for* $h \geq 0$

(ii) $cov\left[\hat{\gamma}(p), \hat{\gamma}(q)\right] = \frac{1}{N}\sum_{r=-(N-p)+1}^{N-q-1}\left[1 - \frac{\eta(r)+q}{N}\right]\{\gamma(r)\gamma(r+q-p) + \gamma(r+q)\gamma(r-p) + \kappa_4(r,p,q-p)\}$

where $\kappa_4(s-t, p, q-p)$ is the fourth joint cumulant of the distribution of $\{x_t, x_{t+p}, x_s, x_{s+q}\}$ and

$$\eta(r) = \begin{cases} r & \text{when } r > 0; \\ 0 & \text{when } -(q-p) \leq r \leq 0; \\ -r-(q-p) & \text{when } -(N-p)+1 \leq r < -(q-p). \end{cases}$$

(iii) For large N, $N cov\left[\hat{\gamma}(p), \hat{\gamma}(q)\right] \sim \sum_{r=-\infty}^{\infty}\{\gamma(r)\gamma(r+q-p) + \gamma(r+q)\gamma(r-p) + \kappa_4(r,p,q-p)\}$

Proof of Lemma 1.2: First part of the lemma is drawn from lemma 1.1. For the second part, we use the usual definition of covariance by considering the case $p, q \geq 0$,

$$\begin{aligned} cov\left[\hat{\gamma}(p), \hat{\gamma}(q)\right] &= E\left[\hat{\gamma}(p)\hat{\gamma}(q)\right] - E\left[\hat{\gamma}(p)\right]E\left[\hat{\gamma}(q)\right] \\ &= \frac{1}{N^2}\sum_{t=1}^{N-p}\sum_{s=1}^{N-q} E(x_t x_{t+p} x_s x_{s+q}) - \left(1 - \frac{p}{N}\right)\left(1 - \frac{q}{N}\right)\gamma(p)\gamma(q) \end{aligned}$$

where

$$\begin{aligned} E(x_t x_{t+p} x_s x_{s+q}) &= E[x_t x_{t+p}]E[x_s x_{s+q}] + E[x_t x_s]E[x_{t+p} x_{s+q}] + E[x_t x_{s+q}]E[x_s x_{t+p}] + \kappa_4 \\ &= \gamma(p)\gamma(q) + \gamma(s-t)\gamma(s+q-t-p) + \gamma(s+q-t)\gamma(s-t-p) + \kappa_4 \end{aligned}$$

and $\kappa_4 = \kappa_4(s-t, p, q-p)$ is the fourth joint cumulant of the joint distribution of $\{x_t, x_{t+p}, x_s, x_{s+q}\}$, which yields

$$cov\,[\hat{\gamma}(p), \hat{\gamma}(q)] = \frac{1}{N^2} \sum_{t=1}^{N-p} \sum_{s=1}^{N-q} \{\gamma(s-t)\gamma(s+q-t-p) + \gamma(s+q-t)\gamma(s-t-p) + \kappa_4(s-t, p, q-p)\}$$

The sum over s, t depends only on their difference in the form of $s-t$. Letting $s-t=r$ and changing variable from s and t to $r = s-t$ and t we can break the double some and can write (Priestley 1981, sec., 5.3.3) in the form as in lemma 1.2.

Since $\frac{\eta(r)+q}{N} \xrightarrow{N} 0$, for large N, $N cov\,[\hat{\gamma}(p), \hat{\gamma}(q)] \sim \sum_{r=-\infty}^{\infty} \{\gamma(r)\gamma(r+q-p) + \gamma(r+q)\gamma(r-p) + \kappa_4(r, p, q-p)\}$.

Lemma 1.3. *If $\{x_t\}$ is a zero mean stationary process and is stationary up to order four then*

(i) $E\hat{\rho}(h) \approx \left(1 - \frac{|h|}{N}\right) \rho(h)$

(ii) $cov\,[\hat{\rho}(p), \hat{\rho}(q)] \approx \frac{1}{\gamma(0)^2} \{cov[\hat{\gamma}(p), \hat{\gamma}(q)] - \rho(p) cov[\hat{\gamma}(0), \hat{\gamma}(p)] - \rho(p) cov[\hat{\gamma}(0), \hat{\gamma}(q)] + \rho(p)\rho(q) var[\hat{\gamma}(0)]\}$

(iii) For large N

$$N cov\,[\hat{\rho}(p), \hat{\rho}(q)] \xrightarrow{N} \sum_{r=-\infty}^{\infty} \{\rho(r)\rho(r+q-p) + \rho(r+q)\rho(r-p) - 2\rho(p)\rho(r)\rho(r-q) - 2\rho(q)\rho(r)\rho(r-p)$$
$$+ 2\rho(p)\rho(q)\rho(r)^2\} + \sum_{r=-\infty}^{\infty} \{\kappa_4(r, p, q-p) + \kappa_4(r, p, -p) + \kappa_4(r, q, -q) + \kappa_4(r, 0, 0)\}$$

where $cov\,[\hat{\gamma}(p), \hat{\gamma}(q)]$ as defined in lemma 1.2.

Proof of Lemma 1.3: Let us define $\delta\{\hat{\gamma}(h)\} = \hat{\gamma}(h) - E\hat{\gamma}(h)$ and $\delta\{\hat{\gamma}(h+k)\} = \hat{\gamma}(h+k) - E\hat{\gamma}(h+k)$. Assuming $\delta\{\hat{\gamma}(h)\}$ and $\delta\{\hat{\gamma}(h+k)\}$ very small (Priestley 1981, eq. 5.3.34) an approximation yields

$$E[\hat{\rho}(h)] \approx \frac{E[\hat{\gamma}(h)]}{E[\hat{\gamma}(0)]} = \left(1 - \frac{|h|}{N}\right) \rho(h)$$

We may write that

$$\begin{aligned} \delta\{\hat{\rho}(p)\} &= \frac{\delta\{\hat{\gamma}(p)\}}{\delta\{\hat{\gamma}(0)\}} \\ &\approx \frac{\hat{\gamma}(0)\delta\{\hat{\gamma}(p)\} - \hat{\gamma}(p)\delta\{\hat{\gamma}(0)\}}{\hat{\gamma}(0)^2} \\ &\sim \frac{\delta\{\hat{\gamma}(p)\}}{\gamma(0)} - \frac{\gamma(p)\delta\{\hat{\gamma}(0)\}}{\gamma(0)^2} \end{aligned}$$

and

$$cov[\hat{\rho}(p), \hat{\rho}(q)] = E[\delta\{\hat{\rho}(p)\}\delta\{\hat{\rho}(q)\}].$$

Thus we may deduce that

$$\begin{aligned} cov[\hat{\rho}(p), \hat{\rho}(q)] \approx &\frac{1}{\gamma(0)^2} \{cov[\hat{\gamma}(p), \hat{\gamma}(q)] - \rho(p) cov[\hat{\gamma}(q), \hat{\gamma}(0)] - \rho(q) cov[\hat{\gamma}(p), \hat{\gamma}(0)] \\ &+ \rho(p)\rho(q) var[\hat{\gamma}(0)]\} \end{aligned}$$

where we may define each term separately and rewrite the above equation as

$$\begin{aligned}\frac{Ncov[\hat{\gamma}(p),\hat{\gamma}(q)]}{\gamma(0)^2} &= \sum_{r=-(N-p)+1}^{N-q-1}\left(1-\frac{\eta(r)+q}{N}\right)\{\rho(r)\rho(r+q-p)+\rho(r+q)\rho(r-p)\\ &\quad +\kappa_4(r,p,q-p)\}\\ &\xrightarrow{N}\sum_{r=-\infty}^{\infty}\{\rho(r)\rho(r+q-p)+\rho(r+q)\rho(r-p)+\kappa_4(r,p,q-p)\}\end{aligned}$$

It is now straightforward to show that

$$\begin{aligned}\frac{Ncov[\hat{\gamma}(0),\hat{\gamma}(p)]}{\gamma(0)^2} &\xrightarrow{N}\sum_{r=-\infty}^{\infty}\{2\rho(r)\rho(r-p)+\kappa_4(r,p,-p)\}\\ \frac{Ncov[\hat{\gamma}(0),\hat{\gamma}(p)]}{\gamma(0)^2} &\xrightarrow{N}\sum_{r=-\infty}^{\infty}\{2\rho(r)\rho(r-q)+\kappa_4(r,q,-q)\}\\ \frac{Nvar[\hat{\gamma}(0)]}{\gamma(0)^2} &\xrightarrow{N}\sum_{r=-\infty}^{\infty}\{2\rho(r)^2+\kappa_4(r,0,0)\}\end{aligned}$$

By adding the all four terms, we get the last part of lemma 1.3.

1.3.2 Asymptotic distribution

Theorem 1.2. *If $\{x_t\}$ is a zero mean stationary process and is stationary up to order four then for any positive integer k,*

$$N^{1/2}\left[\hat{\gamma}(0)-\gamma(0),\hat{\gamma}(1)-\frac{N-1}{N}\gamma(1),\ldots,\hat{\gamma}(k)-\frac{N-k}{N}\gamma(k)\right]\xrightarrow{D}N(\mathbf{0},\mathbf{W}_N)$$

where $\mathbf{W}_N=[W_{p,q}]_{p,q=1,\ldots,k+1}$ and $W_{p+1,q+1}=Ncov\left[\hat{\gamma}(p),\hat{\gamma}(q)\right]$.
For large sample, the bias of the sample autocorrelation function becomes zero and the limiting distribution is

$$N^{1/2}\left[\hat{\gamma}(0)-\gamma(0),\hat{\gamma}(1)-\gamma(1),\ldots,\hat{\gamma}(k)-\gamma(k)\right]\xrightarrow{N}N(\mathbf{0},\mathbf{W})$$

where

$$\begin{aligned}\mathbf{W} &= [w_{p,q}]_{p,q=1,\ldots,k}\\ &\textit{and}\\ w_{p+1,q+1} &= \lim_{N\to\infty}Ncov\left[\hat{\gamma}(p),\hat{\gamma}(q)\right]\\ &= \sum_{r=-\infty}^{\infty}\{\gamma(r)\gamma(r+q-p)+\gamma(r+q)\gamma(r-p)+\kappa_4(s,p,q-p)\}.\end{aligned}$$

Proof of Theorem 1.2: Let $\boldsymbol{\gamma}=[\gamma(0),\gamma(1),\ldots,\gamma(k)]'$ be a vector of first k autocovariances. By lemma 1.2, we may note that

$$E(\hat{\boldsymbol{\gamma}}) = \left[\gamma(0), \frac{N-1}{N}\gamma(1), \ldots, \frac{N-k}{N}\gamma(k)\right]'$$
$$N^{-1}\mathbf{W}_N = E[\{\hat{\boldsymbol{\gamma}} - E\hat{\boldsymbol{\gamma}}\}\{\hat{\boldsymbol{\gamma}} - E\hat{\boldsymbol{\gamma}}\}']$$

where

$$\begin{aligned} W_{p+1,q+1} &= NE[\{\hat{\gamma}(p) - E\hat{\gamma}(p)\}\{\hat{\gamma}(q) - E\hat{\gamma}(q)\}'] \\ &= N cov\,[\hat{\gamma}(p), \hat{\gamma}(q)] \end{aligned}$$

By using CLT we may write

$$N^{1/2}[\hat{\boldsymbol{\gamma}} - E\hat{\boldsymbol{\gamma}}] = N^{1/2}\left[\hat{\gamma}(0) - \gamma(0), \hat{\gamma}(1) - \frac{N-1}{N}\gamma(1), \ldots, \hat{\gamma}(k) - \frac{N-k}{N}\gamma(k)\right] \sim N(\mathbf{0}, \mathbf{W}).$$

For very large N, $E(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}) \xrightarrow{N} \mathbf{0}$ and the above equation yields

$$N^{1/2}\left[\hat{\gamma}(0) - \gamma(0), \hat{\gamma}(1) - \gamma(1), \ldots, \hat{\gamma}(k) - \gamma(k)\right] \xrightarrow{N} N(\mathbf{0}, \mathbf{W})$$

the asymptotic distribution of first k autocovariances.

Theorem 1.3. *If $\{x_t\}$ is a zero mean stationary process and is stationary up to order four then for any positive integer k,*

$$N^{1/2}\left[\hat{\rho}(1) - \frac{N-1}{N}\rho(1), \hat{\rho}(2) - \frac{N-2}{N}\rho(2), \ldots, \hat{\rho}(k) - \frac{N-k}{N}\rho(k)\right] \xrightarrow{D} N(\mathbf{0}, \mathbf{G}_N)$$

where $\mathbf{G}_N = [G_{p,q}]_{p,q=1,\ldots,k+1}$ and $G_{p,q} = N cov\,[\hat{\rho}(p), \hat{\rho}(q)]$.
For large sample, the bias of the sample autocorrelation function becomes zero and the limiting distribution is

$$N^{1/2}\left[\hat{\rho}(1) - \rho(1), \ldots, \hat{\rho}(k) - \rho(k)\right] \xrightarrow{N} N(\mathbf{0}, \mathbf{G})$$

where $\mathbf{G} = [g_{p,q}]_{p,q=1,\ldots,k}$ and $g_{p,q} = \lim_{N\to\infty} N cov\,[\hat{\rho}(p), \hat{\rho}(q)]$ defined in lemma 1.3.

Theorem 1.4. *If $\{x_t\}$ be a general linear process of the form*

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \alpha_j \epsilon_{t-j} \tag{1.2}$$

where ϵ_t are white noise $(0, \sigma^2)$ and $\sum_{j=-\infty}^{\infty} |\alpha_j| < \infty$. Then following result holds

(i) $\sqrt{N}(\bar{x} - \mu) \xrightarrow{N} N\left(0, \gamma(0)\sum_{r=-\infty}^{\infty} \rho(r)\right)$

(ii) if $E\epsilon_t^4 < \infty$ then

$$N^{1/2}\left[\hat{\gamma}(0) - \gamma(0), \hat{\gamma}(1) - \gamma(1), \ldots, \hat{\gamma}(k) - \gamma(k)\right] \xrightarrow{N} N(\mathbf{0}, \mathbf{W})$$

(iii) if $E\epsilon_t^4 < \infty$ and $\sum_{j=-\infty}^{\infty} |j|\alpha_j^2 < \infty$ then

$$N^{1/2}\left[\hat{\rho}(1) - \rho(1), \ldots, \hat{\rho}(k) - \rho(k)\right] \xrightarrow{N} N(\mathbf{0}, \mathbf{G})$$

where $\mathbf{W}$ *and* $\mathbf{G}$ *are as defined before.*

Proof of Theorem 1.4: We note here that

$$E(x_t) = \mu$$
$$E(\bar{x}) = N^{-1} \sum_{t=1}^{N} E(x_t) = N^{-1} N\mu = \mu$$

and variance of the estimator of mean is

$$\begin{aligned} var(\bar{x}) &= var[N^{-1} \sum_{t=1}^{N} x_t] = N^{-2} \sum_{s=1}^{N} \sum_{t=1}^{N} cov[x_s, x_t] \\ &= N^{-2} \sum_{s=1}^{N} \sum_{t=1}^{N} \gamma(t-s) = N^{-2} \gamma(0) \sum_{r=-(N-1)}^{N-1} (N - |r|)\rho(r) \\ &= N^{-1} \gamma(0) \sum_{r=-(N-1)}^{N-1} (1 - |r|/N)\rho(r) \end{aligned}$$

Now that

$$E\left(\sqrt{N}(\bar{x} - \mu)\right) = 0$$

and

$$var[\sqrt{N}(\bar{x} - \mu)] = N var(\bar{x}) \xrightarrow{N} \gamma(0) \sum_{r=-\infty}^{\infty} \rho(r),$$

it follows from above that

$$\sqrt{N}(\bar{x} - \mu) \xrightarrow{N} N(0, \gamma(0) \sum_{r=-\infty}^{\infty} \rho(r)).$$

Now, if $E\epsilon_t^4 < \infty$ then $\{x_t\}$ is stationary up to order four and by lemma 1.2 second part of the theorem holds. Similarly, by lemma 1.3 last part of the theorem can be proved.

Corollary 1.1. *If* ϵ_t *in* (1.2) *is a Gaussian white noise process then* $\kappa_4(r, p, q - p) \equiv 0$ *and*

$$N^{1/2} [\hat{\gamma}(0) - \gamma(0), \hat{\gamma}(1) - \gamma(1), \ldots, \hat{\gamma}(k) - \gamma(k)] \xrightarrow{N} N(\mathbf{0}, \mathbf{W})$$

where $w_{p+1,q+1} = \sum_{r=-\infty}^{\infty} \{\gamma(r)\gamma(r + q - p) + \gamma(r + q)\gamma(r - p)$ *is the* $(p + 1, q + 1)$ *element of* $\mathbf{W}$. *Similarly,*

$$N^{1/2} [\hat{\rho}(1) - \rho(1), \ldots, \hat{\rho}(k) - \rho(k)] \xrightarrow{N} N(\mathbf{0}, \mathbf{G})$$

where (p, q) *element of* $\mathbf{G}$ *is*

$$\begin{aligned} g_{p,q} = \sum_{r=-\infty}^{\infty} &\{\rho(r)\rho(r + q - p) + \rho(r + q)\rho(r - p) \\ &- 2\rho(p)\rho(r)\rho(r - q) - 2\rho(q)\rho(r)\rho(r - p) + 2\rho(p)\rho(q)\rho(r)^2\}. \end{aligned}$$

1.3.2.1 Example 1: White noise process

If $\{x_t\}$ is a white noise process then $\gamma(h) = 0$ for $h \neq 0$. By inserting $\gamma(h > 0) = 0$ in lemma 1.1 the mean of the sample autocovariance function is obtained as

$$E\hat{\gamma}(h) = \begin{cases} -\frac{(N-h)\gamma(0)}{N^2} & \text{if } h > 0; \\ \frac{(N-1)\gamma(0)}{N} & \text{if } h = 0. \end{cases} \tag{1.3}$$

If $\{x_t\}$ is Gaussian white noise process then $\kappa_4(.) \equiv 0$ and from results in lemma 1.2 we may write

$$cov[\hat{\gamma}(p), \hat{\gamma}(q)] = \begin{cases} 0 & \text{if } p \neq q; \\ \frac{(N-p)\gamma(0)^2}{N^2} & \text{if } p = q > 0; \\ \frac{2\gamma(0)^2}{N} & \text{if } p = q = 0. \end{cases} \tag{1.4}$$

Similarly, the mean and variance of sample autocorrelation function are

$$E[\hat{\rho}(h)] \approx \frac{E\hat{\gamma}(h)}{E\hat{\gamma}(0)} = \frac{-(N-h)\gamma(0)}{N^2}\frac{N}{(N-1)\gamma(0)} = -\frac{N-h}{N(N-1)} \tag{1.5}$$

$$cov[\hat{\rho}(p), \hat{\rho}(q)] \approx \frac{cov[\hat{\gamma}(p), \hat{\gamma}(q)]}{\gamma(0)^2} = \begin{cases} 0 & \text{if } p \neq q; \\ \frac{(N-p)}{N^2} & \text{if } p = q > 0. \end{cases} \tag{1.6}$$

Thus for any fixed k,

$$\sqrt{N}\left[\hat{\gamma}(1) + \frac{(N-1)\gamma(0)}{N^2}, \hat{\gamma}(2) + \frac{(N-2)\gamma(0)}{N^2}, \ldots, \hat{\gamma}(k) + \frac{(N-k)\gamma(0)}{N^2}\right]' \sim N(\mathbf{0}, \Omega) \tag{1.7}$$

where $\Omega = \gamma(0)^2 diag\left(\frac{(N-1)}{N}, \frac{(N-2)}{N}, \ldots, \frac{(N-k)}{N}\right)$. Similarly, the distribution of autocorrelation function is

$$\sqrt{N}\left[\hat{\rho}(1) + \frac{(N-1)}{N(N-1)}, \hat{\gamma}(2) + \frac{(N-2)}{N(N-1)}, \ldots, \hat{\gamma}(k) + \frac{(N-k)}{N(N-1)}\right]' \sim N(\mathbf{0}, \gamma(0)^{-2}\Omega) \tag{1.8}$$

1.3.2.2 Example 2: m-dependent process

A stationary sequence $\{x_1, x_2, \ldots\}$ is said to be m-dependent if x_t and x_s are independent, that is, $\gamma(s-t) = 0$ whenever $|s-t| > m$. Let us consider a $MA(m)$ process which is

$$x_t = \epsilon_t + \sum_{j=1}^{m} \epsilon_{t-j} \tag{1.9}$$

where ϵ_t are independently and identically distributed random variables. Since for $MA(m)$ process $\gamma(m+1) = 0$, this is an example of m-dependent process. In this case, the mean and variance of sample autocorrelation function for any $p, q > m$ and $0 \leq |q-p| \leq 2m$, we have $E\hat{\rho}(p) = 0$ and

$$cov[\hat{\rho}(p), \hat{\rho}(q)] = \frac{cov[\hat{\gamma}(p), \hat{\gamma}(q)]}{\gamma(0)^2} = \frac{1}{N}\sum_{r=-m}^{m-|q-p|}\left(1 - \frac{\eta(r) + max(p,q)}{N}\right)\rho(r)\rho(r + |q-p|)$$

where

$$\eta(r) = \begin{cases} r & \text{when } 0 < r < m - |q-p|; \\ 0 & \text{when } -|q-p| \leq r \leq 0; \\ -r - |q-p| & \text{when } -m \leq r < -|q-p|. \end{cases}$$

A large sample approximation of the covariance is

$$N cov[\hat{\rho}(p), \hat{\rho}(q)] \xrightarrow{N} \sum_{r=-m}^{m-|q-p|} \rho(r)\rho(r+|q-p|) \tag{1.10}$$

which yields

$$\sqrt{N}[\hat{\rho}(m+1), \hat{\rho}(m+2), \ldots, \hat{\rho}(m+h)]' \xrightarrow{N} N(\mathbf{0}, \Omega) \tag{1.11}$$

where $\Omega = [\omega_{p,q}]_{p,q=1,\ldots,h}$ and $\omega_{p,q} = \sum_{r=-m}^{m-|q-p|} \rho(r)\rho(r+|q-p|)$.